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ON THE CONNECTIVITY OF RANDOM m-ORIENTABLE GRAPHS AND DIGRAPHS

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We consider graphs and digraphs obtained by randomly generating a prescribed number of arcs incident at each vertex.

We analyse their almost certain connectivity and apply these results to the expected value of random minimum length spanning trees and arborescences.

We also examine the relationship between our results and certain results of Erdős and Rényi.

1. Introduction

This paper is concerned with the connectedness of certain random graphs and digraphs. The most common approach to studying random graphs is either to consider a graph $G_{n,N}$ with n vertices and N edges chosen at random (see Erdős and Rényi [1], [2], [3]), or to fix p, $0<p<1$, and include each possible edge independently with probability p.

However, in his paper on the expected value of a random assignment problem, Walkup [6] uses results on a random bipartite graph that is generated in the following manner: for each vertex v, generate independently at random m distinct edges containing v.

In this paper we consider general graphs generated in an analogous manner. Our motivation for studying these graphs is that various interesting properties of $G_{n,N}$, like connectivity and the existence of hamiltonian cycles, seem to occur when N is large enough to force a lower bound on the degrees of each vertex with probability tending to 1. In this paper we show, for example, that choosing 2 random edges containing v, for each vertex v, is sufficient to ensure connectedness (moreover nonseparability) with probability tending to 1. In a future paper we intend to discuss the existence of hamiltonian cycles in such graphs.*

The structure of the paper is as follows: in Section 2 we consider the vertex and edge connectivities of a random graph $G_m^{(n)}$ with n vertices where, for each vertex

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* We have shown there exists m_0 such that for $m \geqq m_0$ G_m is hamiltonian with probability $\to 1$, [7]. We are currently trying to reduce m_0 (from 23).

v independently, m edges are generated containing v. We show that, with probability tending to 1 as $n\to\infty$, $G_m^{(n)}$ has both vertex and edge connectivities equal to m for $m\geqq 2$, and that $G_1^{(n)}$ is not connected.

In Section 3 we apply these results to establish that the expected length of a minimum spanning tree in the complete n vertex graph, with edge lengths independently drawn from the uniform distribution on $[0, 1]$, never exceeds $2(1+\log n/n)$.

In Section 4 we obtain a characterisation of those graphs which can be instances of $G_m^{(n)}$; this enables us to relate our results to those of Erdős and Rényi [3].

Lastly, in Section 5 we outline similar results for a corresponding class of random digraphs.

2. Connectivity

We begin with the definition of the random digraph $DG_m^{(n)}$, where $1\leqq m\leqq n-1$. The vertex set of $DG_m^{(n)}$ is $V_n=\{1, \dots, n\}$. The arcs of $DG_m^{(n)}$ are obtained by independently taking each $v\in V_n$ and then randomly choosing m distinct arcs (v, w), where $w\in V_n-\{v\}$, so that, for each $v\in V_n$, each of the $\binom{n-1}{m}$ possible sets of arcs has the same probability of being chosen.

$G_m^{(n)}$ is the random graph obtained from $DG_m^{(n)}$ by ignoring the orientation of the arcs. (Strictly, $G_m^{(n)}$ is a multigraph as some edges may occur twice.)

For a graph G, the *vertex connectivity* $C_v(G)$ is the minimum number of vertices the deletion of which disconnects G. The *edge connectivity* $C_e(G)$ is defined similarly in terms of edges.

Theorem 2.1. *(a) For* $m\geqq 2$, $\lim_{n\to\infty} \operatorname{Prob}(C_v(G_m^{(n)})=m)=1$.

(b) For $m\geqq 2$, $\lim_{n\to\infty} \operatorname{Prob}(C_e(G_m^{(n)})=m)=1$.

(c) $\lim_{n\to\infty} \operatorname{Prob}(G_1^{(n)}$ *is connected*$)=0$.

Proof. *(a)* Consider the deletion of p vertices from the random graph $G_m^{(n)}$, where $0\leqq p\leqq m-1$. If $G_m^{(n)}$ can be disconnected by deleting p vertices then there exists a partition (P, S, T) of V_n, where $|P|=p$, $|S|=k$ and $|T|=n-p-k$, with $m-p+1\leqq k\leqq n-m-1$, such that $G_m^{(n)}$ has no edge joining a vertex in S to a vertex in T.

For an arbitrary such partition, the probability of this is

$$\left[\binom{p+k-1}{m}\Big/\binom{n-1}{m}\right]^k\left[\binom{n-k-1}{m}\Big/\binom{n-1}{m}\right]^{n-p-k} \leqq \left(\frac{k+p}{n}\right)^{km}\left(\frac{n-k}{n}\right)^{(n-p-k)m}.$$

Thus,

$$A(m, n, p) = \operatorname{Prob}(C_v(G_m^{(n)}) \leqq p) \leqq \tag{2.1}$$

$$\leqq \sum_{k=m-p+1}^{\lfloor \frac{1}{2}(n-p)\rfloor} \frac{n!}{k!\,p!\,(n-p-k)!}\left(\frac{k+p}{n}\right)^{km}\left(\frac{n-k}{n}\right)^{(n-p-k)m}.$$

Now,

$$\frac{n!}{k!\,p!\,(n-p-k)!} < \frac{12n}{12n-1}\,\frac{n^{n+\frac{1}{2}}}{p!\,e^p k^{k+\frac{1}{2}}(n-p-k)^{n-p-k+\frac{1}{2}}\sqrt{2\pi}} < \frac{\alpha(m, n, p)\,n^n}{k^k(n-p-k)^{n-p-k}},$$

where $\alpha(m, n, p) = \big(12n/(12n-1)\big)\big(n/((m-p+1)(n-m-1))\big)^{\frac{1}{2}}/(p!e^p\sqrt{2\pi})$.

Thus

$$A(m, n, p) \leqq \alpha(m, n, p) \sum_{k=m-p+1}^{\lfloor \frac{1}{2}(n-p) \rfloor} ((k+p)/k)^k((n-k)/(n-p-k))^{n-p-k}u_k$$

where $$u_k = (k+p)^{(m-1)k}(n-k)^{(m-1)(n-p-k)}n^{n-m(n-p)}.$$

Now $((k+p)/k)^k((n-k)/(n-p-k))^{n-p-k} \leqq e^p \times e^p$, and $(k+p)^k(n-k)^{n-p-k}$ decreases monotonically with increasing k for $k \leqq \frac{1}{2}(n-p)$.

Therefore,

$$\text{(2.2)} \qquad A(m, n, p) \leqq \alpha(m, n, p)e^{2p}\big(u_{m-p+1} + \tfrac{1}{2}(n-2m+p-2)u_{m-p+2}\big)$$
$$\leqq \alpha(m, n, p)e^{2p}(an^{1-m(m-p)} + bn^{3-m(m-p+1)})$$

where $a = (m+1)^{(m-1)(m-p+1)}$ and $b = \frac{1}{2}(m+2)^{(m-1)(m-p+2)}$. It follows that $\lim_{n\to\infty} A(m, n, p) = 0$ for $m \geqq 2$ and $0 \leqq p < m$. Thus $\lim_{n\to\infty}$ Prob $\big(C_v(G_m^{(n)}) \geqq m\big) = 1$.

We complete the proof of (a) by showing that

$$B(m, n) = \text{Prob}\,(G_m^{(n)} \text{ has a vertex of degree } m) \to 1 \text{ as } n \to \infty.$$

This will imply that $\lim_{n\to\infty}$ Prob $\big(C_v(G_m) < m\big) = 1$ and hence the result.

Let $E_k^{(n)}$ be the event that vertex k has indegree zero in the associated digraph $DG_m^{(n)}$. Thus the degree of k in $G_m^{(n)}$ is m if and only if $E_k^{(n)}$ occurs.

Now consider $S \subseteqq V_n$ with $|S| = s$. Then

$$q(m, n, s) = \text{Prob}\,\big(\bigcap_{k\in S} E_k^{(n)}\big) = \left(\binom{n-s-1}{m}\Big/\binom{n-1}{m}\right)^{n-s}\left(\binom{n-s}{m}\Big/\binom{n-1}{m}\right)^{s} =$$
$$= \prod_{t=1}^{m} (1-s/(n-t))^n(1+(m/(n-s-m)))^s.$$

Thus, keeping s fixed, we see that $\lim_{n\to\infty} q(m, n, s) = e^{-ms}$. If we let t be a fixed positive integer then, for $n \geqq t$, $B(m, n) \geqq C(m, n, t) = \text{Prob}\left(\bigcup_{k=1}^{t} E_k^{(n)}\right) = \sum_{s=1}^{t} (-1)^{s-1}\binom{t}{s} q(m, n, s)$. Therefore $\lim_{n\to\infty} C(m, n, t) = 1-(1-e^{-m})^t$, so for $n \geqq n_0(t)$, $B(m, n) \geqq 1-(1-e^{-m})^t - 1/t$. However, since t is arbitrary, we see in fact that $\lim_{n\to\infty} B(m, n) = 1$, which completes the proof of (a).

(b) Now $C_e(G_m^{(n)}) \geqq C_v(G_m^{(n)})$ and if $G_m^{(n)}$ has a vertex of degree m then $C_e(G_m^{(n)}) \leqq m$. The result thus follows from (a).

(c) It was shown by Katz [4] that Prob $(G_1^{(n)}$ is connected$) = 2(n-1)! \cdot \left(\sum_{k=0}^{n-2} n^k/k!\right)\Big/(n-1)^n < 2(n-1)!e^n/(n-1)^n \to 0$ as $n \to \infty$. ∎

3. Expected length of a minimum spanning tree

In this section we obtain some results concerning the following question: given a complete graph CG_n with n vertices and with edge lengths drawn independently and uniformly from [0, 1], what is the expected value E_n of the minimum length of a spanning tree of CG_n?

Using an approach derived from that used by Walkup [6] for the assignment problem, we shall show that

$$E_n < 2(1+\log n/n). \tag{3.1}$$

For the moment we consider n as fixed. For each distinct pair i, j with $1 \leqq i, j \leqq n$, let Y_{ij} be a random variable with distribution function

$$F(\lambda) = 1-(1-\lambda)^{\frac{1}{2}}, \quad 0 \leqq \lambda \leqq 1, \tag{3.2}$$

the Y_{ij} all being independent. For $e=\{i,j\}$, if we let $X_e=\min(Y_{ij}, Y_{ji})$ then X_e is uniform in [0, 1].

Walkup observed that $F(\lambda) \geqq H(\lambda)=\lambda/2$ where H is the distribution function for a random variable uniform in [0, 2]. For each i, let $Y_{i,(k)}$ and $U_{(k)}$, respectively, denote the k^{th} smallest of the $n-1$ random variables Y_{ij} and the k^{th} smallest of $n-1$ uniform random variables in [0, 1], then

$$E(Y_{i(k)}) \leqq 2E(U_{(k)}) = 2k/n. \tag{3.3}$$

(In order to overcome the technical difficulty of ties in the definition of the k^{th} smallest of n random variables we delete the tie set, which has probability zero, from the underlying probability space.)

The edge lengths in CG_n are obtained by sampling the Y_{ij} and then computing $X_e=\min(Y_{ij}, Y_{ji})$, as above, for each $e=\{i,j\}$.

Given $\{Y_{ij}\}$ and $m \leqq n-1$, let G_m be the random graph obtained as follows: for each vertex $i=1, 2, \dots, n$, select the m smallest of the Y_{ij}, $1 \leqq j \leqq n$, and then take the m corresponding edges $\{i,j\}$. We note that the graph G_m has the same distribution as the graph $G_m^{(n)}$ of Section 2. The length of edge $e=\{i,j\}$ in G_m is Y_{ij} or Y_{ji} or $\min(Y_{ij}, Y_{ji})$ depending on whether one or other or both of Y_{ij} and Y_{ji} are selected.

For a given Y we obtain a spanning tree T of CG_n as follows:

1. Construct G_2.
2. If G_2 is connected then
  (a) Construct G_1 and delete the longest edge from any cycle.
  (b) Connect up the remaining forest using edges of G_2-G_1.
3. If G_2 is not connected then choose an arbitrary spanning tree in CG_n.

Let F_n be the expected length of T and, for $1 \leqq k \leqq [n/2]$, let $q_k=$ Prob(G_1 has k components and G_2 is connected). Let $\pi_n=$ Prob(G_2 is not connected)$=A(2, n, 0)$.

Now

$$E_n \leqq F_n < \sum_{k=1}^{\lfloor n/2 \rfloor} (2(n-k)/n+4(k-1)/n)q_k+(n-1)\pi_n.$$

The summation corresponds to the case when G_2 is connected because if G_1 has k components then T uses $n-k$ edges of G_1, whose expected lengths do not exceed $2/n$, by (3.3), and $k-1$ edges of G_2-G_1, whose expected lengths do not exceed $4/n$. The term $(n-1)\pi_n$ corresponds to the case where G_2 is not connected, in which case the length of T does not exceed $n-1$.

We note that $\pi_n=0$ for $n\leqq 5$, and from (2.2) it is straightforward to show that

$$\pi_n \leqq 1/n(n-1) \quad \text{for} \quad n \geqq 2. \tag{3.4}$$

Now $q_k\leqq p_k=\text{Prob}\,(G_1$ has k components) and thus

$$E_n < \sum_{k=1}^{\lfloor n/2 \rfloor} \big(2(n-k)/n+4(k-1)/n\big)p_k+(n-1)\pi_n =$$
$$= 2-4/n+2\sum_{k=1}^{\lfloor n/2 \rfloor} kp_k/n+(n-1)\pi_n < 2\left(1+\sum_{k=1}^{\lfloor n/2 \rfloor} kp_k/n\right), \quad \text{by (3.4).}$$

Now Kruskal [5] shows that

$$\sum_{k=1}^{\lfloor n/2 \rfloor} kp_k \leqq \sum_{j=1}^{n} n!/\big((n-j)!\,jn^j\big) < \log n$$

(3.1) now follows immediately.

It seems reasonable to conjecture that E_n is monotonically increasing in n, in which case (3.1) would imply $E_n\leqq 2$ for all n.

4. m-orientable graphs

Every instance of the graph $G_m^{(n)}$ has n vertices and mn edges where an edge can be repeated at most twice and each vertex has degree at least m. It is not true, however, that these properties characterise $G_m^{(n)}$. In this section, therefore, we obtain such a characterisation.

We first consider graphs without repeated edges. Given a graph $G=(V, E)$, an *orientation* $\Omega(G)$ of G is a digraph $\big(V, \Omega(E)\big)$ in which

$$(v, w)\in\Omega(E) \Rightarrow \{v, w\}\in E, \tag{4.1a}$$

$$\{v, w\}\in E \Rightarrow \big|\{\Omega(E)\cap\{(v, w),\ (w, v)\}\big| = 1, \tag{4.1b}$$

i.e. each edge of E is oriented by Ω.

Now let α: $V\to Z^+$, the set of non-negative integers. The graph G is α-*orientable* if there is some orientation $\Omega(G)$ for which the outdegree of v is at least $\alpha(v)$ for all $v\in V$. (Our aim is to find a way of determining whether a graph G is a $G_m^{(n)}$, i.e. whether it can be oriented to yield a $DG_m^{(n)}$.)

For a set $S\subseteqq V$, we let $\alpha(S)=\sum_{v\in S}\alpha(v)$. We now obtain a characterisation of α-orientability.

Theorem 4.1.* *A graph $G=(V, E)$ is α-orientable if and only if, for all $S\subseteqq V$,*

$$\big|\{e\in E\colon\ e\cap S \neq \emptyset\}\big| \geqq \alpha(S). \tag{4.1}$$

* This result was also obtained by Frank and Gyárfás [8].

Proof. The necessity of 4.1 is evident, for if $\Omega(G)$ is an α-orientation then at least $\alpha(S)$ arcs must have their tails in S.

To prove sufficiency, we consider the following maximum flow problem (see Figure 1):

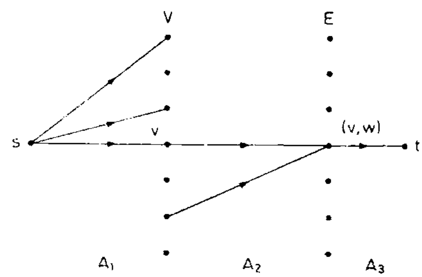

Fig. 1

Let D be the digraph (X, A) where $X=\{s, t\}\cup V\cup E$, $A=A_1\cup A_2\cup A_3$,

$$A_1 = \{(s, v):\ v\in V\},$$

$$A_2 = \{(v, e):\ v\in e\in E\},$$

$$A_3 = \{(e, t):\ e\in E\}.$$

The flow capacities $c\colon A\to Z^+$ are defined by

$$c(a) = \begin{cases} \alpha(v) & \text{if} \quad a=(s, v)\in A_1, \\ \infty & \text{if} \quad a\in A_2, \\ 1 & \text{if} \quad a\in A_3. \end{cases}$$

We observe first that G is α-orientable if and only if the maximum value of an $s-t$ flow in D is $\alpha(V)$.

Condition (4.1) ensures that a cut separating s and t which contains no arcs of A_2 has capacity at least $\alpha(V)$. Then, on applying the Max-Flow-Min-Cut Theorem of Ford and Fulkerson, we obtain the result. ∎

In order to deal with multiple edges, we now define a graph as $G=(V, E, \mu)$ where V, E are as before and $\mu\colon E\to\{1, 2\}$ gives the edge multiplicities. An edge $e=\{v, w\}$ with $\mu(e)=2$ should be oriented in both directions to correspond with the way double edges arise in obtaining $G_m^{(n)}$ from $DG_m^{(n)}$. We therefore change the definition of an orientation, replacing (4.1b) by

(4.1b′) $$\{v, w\}\in E \Rightarrow |\Omega(E)\cap\{(v, w),\ (w, v)\}| = \mu(e).$$

The definition of α-orientable remains unchanged. It now follows easily that, if $e\in E$ and $\mu(e)=2$, $G=(V, E, \mu)$ is α-orientable if and only if $G'=(V, E-\{e\}, \mu')$ is α'-orientable, where $\alpha'(v)=\alpha(v)$ for $v\notin e$ and $\alpha'(v)=\max\big(0, \alpha(v)-1\big)$ for $v\in e$, and μ' is the restriction of μ to $E-\{e\}$. On combining this remark with Theorem 4.1 we obtain

Corollary 4.1. *A graph $G=(V, E, \mu)$ is α-orientable if and only if, for all $S \subseteqq V$,*

(4.2) $$|\{e \in E:\ e \cap S \neq \emptyset\}| + \sum_{e \subseteqq S} (\mu(e)-1) \geqq \alpha(S). \quad \blacksquare$$

If m is a positive integer, we use the term m-orientable to denote c_m-orientable where $c_m(v)=m$ for all $v \in V$. Hence we see that the graphs $G_m^{(n)}$ generated as in Section 2 are precisely the m-orientable graphs for which $\sum_{e \in E} \mu(e)=mn$.

Erdős and Rényi [3] have shown that, for a fixed integer $m \geqq 0$ and real c, if $N=N(n)=\frac{1}{2}n\log n+\frac{1}{2}mn\log\log n+cn$ and $G_{n,N}$ is a random graph with n vertices and N edges then

(4.3) $$\lim_{n\to\infty} \operatorname{Prob}(C_v(G_{n,N}) = m) = 1-\exp(-e^{-2c}/m!).$$

This is derived by showing that, with probability tending to 1, $C_v(G_{n,N}) \geqq m$ and with probability tending to $1-\exp(-e^{-2c}/m!)$ $G_{n,N}$ has a vertex of degree m.

We next show that

(4.4) $$\lim_{n\to\infty} \operatorname{Prob}(G_{n,N} \textit{ is m-orientable}) = 1,$$

which combined with a modification of Theorem 2.1 for the case with no repeated edges, leads us to conjecture that the m-connectivity of $G_{n,N}$ could be deduced from its almost certain m-orientability, provided $m \geqq 2$.

Now if $G_{n,N}$ is not m-orientable, by Theorem 4.1 there exists a set S of s vertices which meet k edges, where $k \leqq ms-1$. Thus $\operatorname{Prob}(G_{n,N} \textit{ is not m-orientable}) \leqq \Delta_n$, where

(4.5) $$\Delta_n = \sum_{s=1}^{n} \binom{n}{s} \sum_{k=0}^{ms-1} \binom{\binom{n}{2}-\binom{n-s}{2}}{k} \binom{\binom{n-s}{2}}{N-k} \Big/ \binom{\binom{n}{2}}{N}$$

and we take $\binom{\alpha}{\beta}=0$ in the above summation if $\beta<0$ or $\alpha<\beta$, in order to simplify the range of summation. We show in an appendix, that $\lim_{n\to\infty} \Delta_n=0$, which yields the desired result (4.4).

5. Digraphs

Our results on the connectivity of $G_m^{(n)}$ extend easily to digraphs. The random digraph $D_m^{(n)}$ has vertex set V_n. The arcs of $D_m^{(n)}$ are obtained by first independently taking each $v \in V_n$, then randomly choosing m distinct arcs (v, w) where $w \in V_n-\{v\}$ and finally randomly choosing another m distinct arcs (w', v) where $w' \in V_n-\{v\}$.

One can easily show that, if $\hat{A}(m, n, p)$ is the probability that there exists a set of p vertices whose deletion from $D_m^{(n)}$ leaves the remaining digraph not strongly connected, and if $\hat{B}(m, n)$ is the probability that $D_m^{(n)}$ contains a vertex with outdegree m, then

(5.1a) $$\hat{A}(m, n, p) \leqq 2\times[\text{R.H.S. of (2.1)}],$$

(5.1b) $$\hat{B}(m, n) = B(m, n).$$

We thus have

Theorem 5.1. *(a) For* $m\geqq 2$, $\lim_{n\to\infty} \text{Prob}\left(SC_v(D_m^{(n)})=m\right)=1$,

(b) For $m\geqq 2$, $\lim_{n\to\infty} \text{Prob}\left(SC_e(D_m^{(n)})=m\right)=1$

where, for any digraph D, $SC_v(D)$ *is the minimum number of vertices that must be deleted in order that the remaining digraph be not strongly connected.* $SC_e(D)$ *is defined similarly in terms of arcs (directed edges).* ▮

It is at present not known whether or not the probability that $D_1^{(n)}$ is strongly connected tends to 1.

Minimum length arborescences

The results on the expected length of a minimum spanning tree can be partially generalised to the expected length of a minimum spanning arborescence rooted at vertex 1. (An arborescence rooted at vertex 1 is a digraph in which vertex 1 has indegree 0, every other vertex has indegree 1, and the graph obtained by ignoring arc orientations is a tree.)

Assuming that we are given a complete digraph CD_n with n vertices and with arc lengths drawn independently and uniformly from [0, 1], we shall show that the expected length $\hat{E}_n$ of a minimum spanning arborescence rooted at vertex 1 is less than 3.

For each distinct pair i, j with $1\leqq i, j\leqq n$, let Y_{ij} and Z_{ij} be independent random variables with distribution function (3.2). Let $X_{ij}=\min(Y_{ij}, Z_{ij})$ be the length of the arc (i, j) in CD_n. Let D_m be the random digraph obtained as follows: for each vertex $i=1, 2, \ldots, n$, select the m smallest of the Y_{ij}, $1\leqq j\leqq n$, and then take corresponding arcs (i, j) and similarly select the m smallest of the Z_{ji} and take the corresponding arcs (j, i). We see that the digraph D_m has the same distribution as the digraph $D_m^{(n)}$.

For given Y and Z, we construct a spanning arborescence R as follows: if D_2 is strongly connected then it contains at least one spanning arborescence rooted at vertex 1 so we choose the shortest of these, and if D_2 is not strongly connected then we choose an arbitrary spanning arborescence in CD_n.

Let $\hat{F}_n$ be the expected length of R, then we see that

$$\hat{E}_n \leqq \hat{F}_n \leqq 3(n-1)/n+(n-1)\hat{\pi}_n$$

where $\hat{\pi}_n=\text{Prob}\,(D_2$ *is not strongly connected*$)=\hat{A}(2, n, 0)$. The term $3(n-1)/n$, which corresponds to the case when D_2 is strongly connected, is $n-1$ times the expected length of an arc of D_2.

The result now follows as $3/n>(n-1)\hat{\pi}_n$ since $\hat{\pi}_n\leqq 2\pi_n\leqq 2/n(n-1)$ by (5.1) and (3.4).

As in Section 4 we pursue a characterisation of the digraphs $D_m^{(n)}$. This time, given α, β: $V\to Z^+$ and a digraph $D=(V, A)$, we determine whether there is a function $\varphi\colon A\to V$ such that

(5.2a) $$\varphi((v, w))\in\{v, w\} \quad \text{for} \quad (v, w)\in A,$$

(5.2b) $$|\{(v, w)\in A\colon \varphi((v, w)) = v\}| \geqq \alpha(v) \quad \text{for} \quad v\in V,$$
$$|\{(w, v)\in A\colon \varphi((w, v)) = v\}| \geqq \beta(v) \quad \text{for} \quad v\in V.$$

The digraphs $D_m^{(n)}$ without repeated arcs satisfy (5.2) with $\alpha=\beta=c_m$, where $c_m(v)=m$ for $v\in V$.

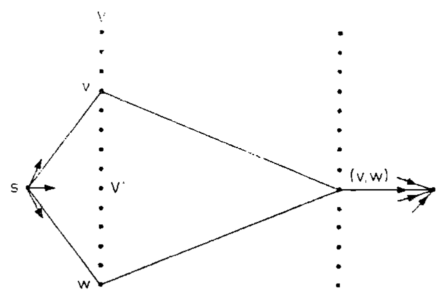

Fig. 2

By considering the maximum flow problem (see figure 2):

Vertices: $\{s, t\}\cup V\cup V'\cup A$, where $V' = \{v'\colon v\in V\}$

Arcs: $A_1\cup A_2\cup A_3$

$A_1 = \{(s, v)\colon v\in V\}\cup\{(s, v')\colon v'\in V\}$, Capacity: $c(s, v) = \alpha(v)$

$c(s, v') = \beta(v)$

$A_2 = \bigcup_{(v, w)\in A} \{(v, a),\ (w', a)\}$, Capacity: $\infty\,(\geqq \alpha(V)+\beta(V)+1)$

$A_3 = \{(a, t)\colon a\in A\}$, Capacity: 1, we obtain

Theorem 5.2. *Given α and β, there exists φ satisfying* (5.2) *if and only if for all $S, T\subseteq V$ we have*

(5.3) $$|(v, w)\in A\colon v\in S \quad \text{or} \quad w\in T\}| \geqq \alpha(S)+\beta(T). \quad \blacksquare$$

Repeated arcs are dealt with as in section 4, i.e. by considering the reduced problem obtained by ignoring repetitions and modifying α and β appropriately. This leads to an extra term $\sum(\mu(v, w)-1)$ on the left side of (5.3), where the summation is over all $(v, w)\in A$ with $v\in S$ and $w\in T$.

Appendix

We first define U_{sk} for $1 \leqq s \leqq n$ and $0 \leqq k \leqq ms-1$ by

$$U_{sk} = \binom{\binom{n}{2}-\binom{n-s}{2}}{k}\binom{\binom{n-s}{2}}{N-k} \quad \text{if} \quad k \leqq \binom{n}{2}-\binom{n-s}{2} \quad \text{and} \quad N-k \geqq \binom{n-s}{2}$$

$$U_{sk} = 0 \qquad \text{otherwise.}$$

We show first that if $U_{s,k+1} \neq 0$ then $U_{s,k+1} \geqq U_{sk}$ provided n is large enough. Now

$$\frac{U_{s,k+1}}{U_{sk}} = \frac{\left(\binom{n}{2}-\binom{n-s}{2}-k\right)(N-k)}{(k+1)\left(\binom{n-s}{2}-N+k+1\right)}$$

and, for large n,

$$\left(\binom{n}{2}-\binom{n-s}{2}-k\right)(N-k)-(k+1)\left(\binom{n-s}{2}-N+k+1\right)$$

$$= N\left(\binom{n}{2}-\binom{n-s}{2}\right)-\left(k\binom{n}{2}+\binom{n-s}{2}\right)+N-2k-1$$

$$\geqq N\left(s(n-s)+\tfrac{1}{2}s(s-1)\right)-ms\binom{n}{2} = s\left(N(n-\tfrac{1}{2}s-\tfrac{1}{2})-\tfrac{1}{2}mn(n-1)\right) \geqq 0$$

Now for large n we have $\binom{n}{2}-\binom{n-s}{2} \geqq ms$. Thus if $a(n)=\max\left(s\colon \binom{n-s}{2} \geqq \right.$ $\left.\geqq N-ms+1\right)$ we see by the above that

$$\Delta_n \leqq \sum_{s=1}^{a(n)} v_s \Bigg/ \binom{\binom{n}{2}}{N}$$

where

$$v_s = \binom{n}{s} ms \binom{\binom{n}{2}-\binom{n-s}{2}}{ms-1}\binom{\binom{n-s}{2}}{N-ms+1}.$$

Now let

$$\Delta'_n = \sum_{s=1}^{n/2} v_s \Big/ \binom{\binom{n}{2}}{N} \leqq \sum_{s=1}^{n/2} \frac{n^s}{s!}\, ms \binom{\binom{n}{2}-\binom{n-s}{2}}{ms-1} PQ$$

where

$$P = \binom{\binom{n-s}{2}}{N-ms+1} \Big/ \binom{\binom{n-s}{2}}{N} \leqq \left(\frac{N}{\binom{n-s}{2}-N}\right)^{ms-1} \leqq (\alpha \log n/n)^{ms-1}$$

for some $\alpha > 0$ when n is large, and where

$$Q = \binom{\binom{n-s}{2}}{N} \Big/ \binom{\binom{n}{2}}{N} = \prod_{t=0}^{N-1} \left(\frac{\binom{n-s}{2}-t}{\binom{n}{2}-t}\right) \leqq \left[\frac{\binom{n-s}{2}}{\binom{n}{2}}\right]^N \leqq \left(1-\frac{s}{n}\right)^{2N}$$

$$\leqq \exp(-s\log n - sm \log\log n - 2cs) = n^{-s}(\log n)^{-ms} e^{-2cs}.$$

Also

$$\binom{\binom{n}{2}-\binom{n-s}{2}}{ms-1} \leqq \frac{\left(s(n-\frac{1}{2}s-\frac{1}{2})\right)^{ms-1}}{(ms-1)!} \leqq (2en)^{ms-1}$$

and so

$$\Delta'_n \leqq \sum_{s=1}^{\infty} \frac{m}{(s-1)!} (2ae)^{ms-1} \frac{e^{-2cs}}{\log n} = O\left(\frac{1}{\log n}\right)$$

and so $\Delta'_n \to 0$.

Now let $\Delta''_n = \sum_{s=\frac{n}{2}+1}^{a(n)} v_s \Big/ \binom{\binom{n}{2}}{N}$. Now

$$\frac{v_{s+1}}{v_s} = \frac{n-s}{s} UV$$

where

$$U = \prod_{t=0}^{m-1} \left[\frac{\binom{n}{2}-\binom{n-s}{2}-m(s+1)+t+1}{\binom{n-s}{2}-(N-m(s+1)+t+1)}\right] \left[\frac{N-m(s+1)+t+2}{m(s+1)-t-1}\right]$$

$$\geqq \left[\frac{\left(\binom{n}{2}-\binom{n-s}{2}-m(s+1)\right)(N-m(s+1)+2)}{\binom{n-s}{2} m(s+1)}\right]^m \geqq \alpha \left(\frac{s\log n}{n-s}\right)^m$$

for some $\alpha > 0$ when n is large and

$$V = \prod_{t=0}^{n-s-2} \left[\frac{\binom{n}{2}-\binom{n-s}{2}+t-m(s+1)+1}{\binom{n}{2}-\binom{n-s}{2}+t+1}\right]\left[\frac{\binom{n-s}{2}-t}{\binom{n-s}{2}-t-(N-ms)}\right]$$

$$\geqq \left[\frac{\binom{n}{2}-\binom{n-s}{2}-m(s+1)+1}{\binom{n}{2}-\binom{n-s}{2}+n-s-1}\right]^{n-s-2} \geqq \beta$$

for some $\beta > 0$ when n is large.
Thus

$$\frac{v_{s+1}}{v_s} \geqq \alpha\beta\left(\frac{s}{n-s}\right)^{m-1} (\log n)^m \geqq \alpha\beta (\log n)^m \geqq 1 \quad \text{as } s \geqq \frac{n}{2},$$

and this is $\geqq 1$, for large n.

Thus

(A1) $$\Delta_n'' \leqq \tfrac{1}{2} n v_{a(n)} \Bigg/ \binom{\binom{n}{2}}{N}.$$

From its definition $a = a(n)$ satisfies

(A2) $$\binom{n-a-1}{2} < N-m(a+1)+1 \text{ or}$$
$$\binom{n-a}{2}-(n-a-1) < (N-ma+1)-m$$

from which we deduce $\binom{n-a}{2}-(N-ma+1) < n-a.$

From (A2) we can also deduce that

$$(n-a)^2-3(n-a)+2 < 2(N-m(a+1)+1)$$

from which we obtain $n-a \leqq \sqrt{2N}$ assuming n large.
It then follows that

$$\binom{\binom{n-a}{2}}{N-ma+1} \leqq \binom{\binom{n-a}{2}}{n-a} \leqq \binom{N}{\sqrt{2N}} \quad \text{as} \quad n-a \leqq \tfrac{1}{2}\binom{n-a}{2} \quad \text{for large } n$$

and so

$$v_a \leqq \binom{n}{a} ma \binom{\binom{n}{2}-\binom{n-a}{2}}{ma-1} \binom{N}{\sqrt{2N}}.$$

It follows easily from (A1), since $\binom{\binom{n}{2}}{N}$ is much larger than nv_a, that $\Delta_n'' \to 0$. Thus $\Delta_n \to 0$ as was to be shown.

Note added in proof. A. M. Frieze has recently improved the result of Section 3 by showing that $E_n \to \zeta(3)$.

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